JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **54**, 846–848 (1976)

On "Thermal Instability in a Viscoelastic Fluid Layer in Hydromagnetics"

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In a paper by Bhatia and Steiner [1], the convective instability of a horizontal layer of a Maxwellian fluid heated from below was studied in the presence of a uniform magnetic field. The critical mode of instability was located for different values of the Prandtl number p_1, the magnetic Prandtl number p_2, and the Chandrasekhar number Q. However, the solutions obtained do not satisfy the correct boundary conditions. The discrepancy is due to the boundary condition on the vertical component of the perturbation magnetic field K. The general boundary condition for the continuity of K has been shown by Gibson [4] to be

$$dK/dz \pm \Omega K = 0 \qquad \text{at} \qquad z = \pm\tfrac{1}{2}, \tag{1}$$

with

$$\Omega^2 = a^2 + p_m\sigma, \tag{2}$$

where the lower and upper boundaries are here located at $z = \pm\frac{1}{2}$. Here a is the dimensionless wavenumber, σ is the growth rate, and p_m $(= \nu/\eta)$ is the magnetic Prandtl number of the boundary based on the kinematic viscosity ν of the fluid and the magnetic diffusivity η of the boundary. We may note that condition (1) reduces to $K = 0$ *only* if $p_m = \infty$, i.e., the boundary is a perfect conductor. For finite conductivity of the boundary, however, condition (1) must be applied in its entirety. When $\sigma = 0$ or $p_2 = 0$, this condition may be ignored as far as the eigenvalue problem for R_c is concerned but must be applied if K is required [3]. We should also point out here that the conditions applied by Chandrasekhar [2] for the inelastic layer are justified in those particular cases considered [6].

The above discussion indicates that the solution of Bhatia and Steiner applies only if the bounding surfaces are perfect electrical conductors. Furthermore, their numerical calculations of the critical mode show that overstability is preferred for large values of Q $(\simeq 10^6)$ when $p_2 < p_1$. This, as we shall presently see, may not be the case.

In this note, we extend Gibson's asymptotic technique to the present problem. This method is effective only for large values of Q. However, it yields analytic solutions which facilitate direct comparison with the well-studied and related problem of the inelastic layer. It also illustrates, very clearly, the dependence of the eigenvalue problem on the boundary conditions.

Since the basic equations for this problem are the same as those of the inelastic layer, apart from replacing ν by $\nu/(1 + \sigma\Gamma)$, we shall omit the details of the method here. Γ is the relaxation time. The critical mode for free boundaries is given by (using the notation of Bhatia and Steiner)

$$R_c = \frac{p_1^2(1+\alpha p_2)\pi^2 Q}{p_2^2(1+\alpha p_1)}\left[1 + \frac{3(1+\alpha p)(p_1+p_2)a_c^4}{\pi^2 p_1^2 Q}\right],$$
$$\sigma_c^2 = \frac{\pi^2(p_2-p_1)Q}{p_1^2(1+\alpha p_1)}, \qquad (p_2 > p_1), \quad Q \to \infty. \tag{3}$$

The wavenumber at marginal stability depends crucially on the electrical conductivity of the boundary. Thus

$$a_c^{24} = \frac{4\pi^{10}p_1^8(p_2-p_1)Q^5}{p_m^2(1+\alpha p_2)^4(1+\alpha p_1)^5(p_2+p_1)^4}, \qquad Q \to \infty, \tag{4}$$

when the boundary has finite conductivity;

$$a_c^5 = \frac{4p_1^2(2+\alpha p_2)\pi^2 Q}{(1+p_1\alpha)(1+\alpha p_2)(p_1+p_2)}, \qquad Q \to \infty, \tag{5}$$

when the boundary is a perfect conductor; and

$$a_c^5 = \frac{\pi^2(2+\alpha p_1)p_1^2 Q}{2(1+\alpha p_2)(1+\alpha p_1)(p_1+p_2)}, \qquad Q \to \infty, \tag{6}$$

in the case of an insulating boundary. Here α is defined by

$$\begin{aligned} \alpha &= 1 \qquad \text{for Newtonian fluid,} \\ &= 0 \qquad \text{for Maxwellian fluid.} \end{aligned} \tag{7}$$

In fact these results apply for fluids with nonzero retardation time Γ_1 [5] if we define α as Γ_1/Γ. The term involving a_c^4 in the expression for R_c is small compared to unity but is included to calculate the error involved.

The expression for σ for the Newtonian layer (see [2, p. 183, Eq. (230)]) shows that overstable motions exist only if $p_2 > p_1$, for all values of Q. Since overstable motions are preferred for $Q = 0$ for the elastic layer, it may be deduced that overstable modes may exist for $p_2 < p_1$ if Q is small. As Q increases, overstable modes will stay preferred as long as Q is less than a certain value Q_0. If Q increases beyond Q_0, overstable modes will give the preferred mode only if $p_2 > p_1$ as equation (2) above shows. If $p_2 \leqslant p_1$, stationary convection yields the critical mode, as in the case of the inelastic layer.

These results can be extended to the case of rigid boundaries if we replace every term of the form $x + p_1$ or $x + p_2$, when x is a real number, in the expressions obtained for the inelastic layer [4] by $x + \alpha p_1$ and $x + \alpha p_2$, respectively.

It is now clear that the fluid layer is destabilized by the presence of elasticity since $\alpha = 0$ yields a smaller R_c than $\alpha = 1$. Furthermore, the destabilizing influence of elasticity can be traced down to the fact that elasticity ***inhibits*** viscous dissipation of energy while all other forms of energy are unaffected by it. If V is the rate of viscous dissipation of energy for the inelastic layer, it can be shown that it is $\bar{V}$ for the elastic layer where

$$\bar{V} = V/(1 + \sigma\Gamma).$$

When $|\sigma| \gg 1$, as in Eq. (3) above, $\bar{V} \ll V$.

References

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